

Calculus I Practice Final Exam

This practice test features cumulative problems based on the content in the following chapters.

[Chapter 1: Limits and Continuity](#)

[Chapter 2: Differentiation Rules](#)

[Chapter 3: Applications of Differentiation](#)

[Chapter 4: Integration](#)

(Click on a chapter to open the online textbook.)

Directions: For each question, show your work. Follow these guidelines:

- Do not use a calculator of any kind.
- Adhere to the time limit of 180 minutes.
- After you complete all the questions, score yourself according to the Solutions at the end.
Note any topics that require revision.

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Calculus I**Number of Questions—25****Suggested Time—3 hours****NO CALCULATOR****Scoring Chart**

Section	Points	Points Available
True/False Questions		22
Short Questions		33
Question 23		15
Question 24		15
Question 25		15
TOTAL		100

True/False Questions

Circle "TRUE" or "FALSE" for each statement.

1. If f has a jump discontinuity at a , then $\lim_{x \rightarrow a} f(x)$ does not exist. (2 pts.)

TRUE FALSE

2. The limit $\lim_{x \rightarrow 1} \frac{\sin(x^2 - 1)}{x^2 - 1}$ exists. (2 pts.)

TRUE FALSE

3. If $f(t) \geq g(t)$ on $[a, b]$, then $f'(t) \geq g'(t)$ on $[a, b]$. (2 pts.)

TRUE FALSE

4. The Mean Value Theorem applies to $h(x) = \frac{1}{5}|10 - 2x| - 1$ on $[4, 8]$. (2 pts.)

TRUE FALSE

5. If $f'(c) = 0$, then f has a relative extremum at c . (2 pts.)

TRUE FALSE

6. All odd, integrable functions f on $[-a, a]$ satisfy $\int_{-a}^0 f(x) \, dx - \int_0^a f(-x) \, dx = 0$. (2 pts.)

TRUE FALSE

7. $\int_{-2}^2 \frac{1}{x^4} \, dx = \frac{x^{-3}}{-3} \Big|_{-2}^2$. (2 pts.)

TRUE FALSE

8. For $g(x) = 2\sin(4x - \pi) - 1$, the n th derivative $g^{(n)}(x)$ has an amplitude 4^n times the amplitude of $g(x)$. (2 pts.)

TRUE

FALSE

9. The Trapezoidal Rule overestimates the area under a curve that is concave up. (2 pts.)

TRUE

FALSE

10. Newton's Method tends to increase in accuracy in estimating a zero when the initial guess is selected near a horizontal tangent. (2 pts.)

TRUE

FALSE

11. By substituting $u = e^x + 2$, $\int_{\ln 2}^{\ln 4} \frac{e^x}{e^x + 2} dx = \int_4^6 \frac{u-2}{u} du$. (2 pts.)

TRUE

FALSE

Short Questions

12. Find all the horizontal asymptotes and vertical asymptotes of the graph of $y = \frac{4x^3}{x^3 - x}$. (3 pts.)

13. For $f(x) = \sinh^2 x - \tan^{-1} 2x$, determine $f'(1)$. (3 pts.)

14. Calculate the values of p and q such that the function defined by

(3 pts.)

$$g(x) = \begin{cases} -\frac{4}{x^2 + p} & x \geq 1 \\ \frac{1}{2}(2x - 1)^2 + q & x < 1, \end{cases}$$

where $p > 0$, is differentiable at $x = 1$.

15. Determine the interval on which the function $p(x) = 2x^{5/2} - 10x^{3/2}$ is decreasing and concave up. (3 pts.)

16. Use Simpson's Rule with $n = 4$ to approximate $\int_1^9 \frac{2}{\sqrt{x}} dx$. (3 pts.)

17. Evaluate $\lim_{t \rightarrow \infty} \left(\frac{9t - 4t^2}{1 - 4t^2} \right)^t$.

(3 pts.)

18. A food stand sells 20 quesadillas per night when they are priced at \$4 each. The manager notes that 4 fewer quesadillas are sold for every increase of \$2 in the quesadilla's price. Calculate the food stand's maximum possible revenue from quesadillas.

(3 pts.)

19. Find the exact value of $\int_0^7 \frac{x}{\sqrt[3]{x+1}} dx$.

(3 pts.)

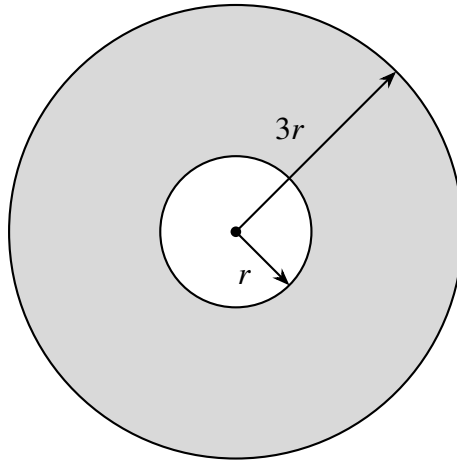
20. The acceleration of a block attached to a spring as a function of time $t \geq 0$ is given by $a(t) = 4 \sin 2t + 8 \cos 2t$, where $a(t)$ is measured in meters per second per second and t is measured in seconds. Initially, the block is stationary and has a displacement of 1 meter. Find the block's displacement as a function of time.

(3 pts.)

21. Evaluate $\frac{d}{dx} \left(\frac{e^{-x} \sqrt{2x+7} \sec^6 4x}{x^4 \tan(2x+9)} \right)$.

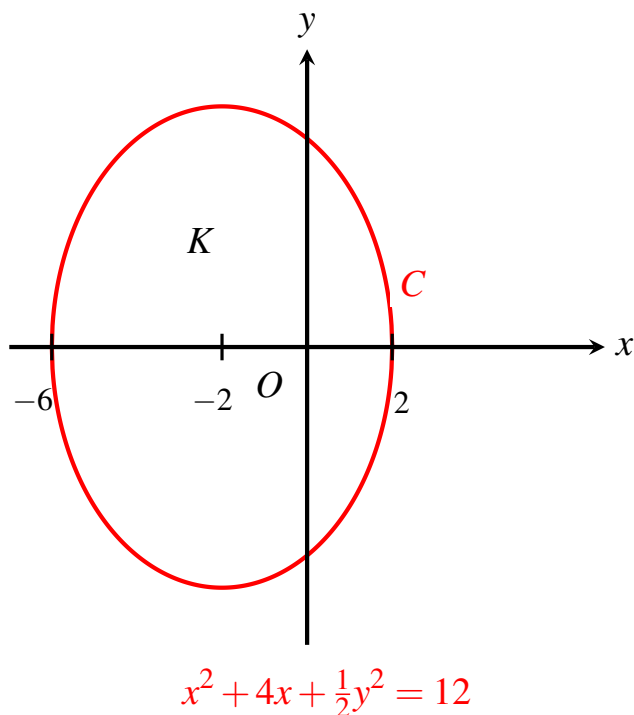
(3 pts.)

22. The figure below shows two concentric circles of radii r and $3r$. Using differentials, approximate the amount by which the area between the circles increases when r is increased from 2 to 2.1 inches. (3 pts.)



Long Questions

23. The implicit equation $x^2 + 4x + \frac{1}{2}y^2 = 12$ produces the curve C shown below. Let K be the region in the second quadrant bounded by the curve, the x -axis, and the y -axis.



- (a) Using Implicit Differentiation, show that $\frac{dy}{dx} = \frac{-2x - 4}{y}$.

(2 pts.)

- (b) Write equations for the line tangent to C and the line normal to C at the point $(1, \sqrt{14})$.

(2 pts.)

- (c) For the curve C , find $\frac{d^2y}{dx^2}$ in terms of x and y .

(3 pts.)

- (d) Write a definite integral that equals the area of region K . Use a left-endpoint approximation with $n = 3$ to approximate the value of this integral.

(3 pts.)

- (e) A rectangle is inscribed in the curve C . Calculate the width of the rectangle with the maximum possible area.

(5 pts.)

24. In a building, a mouse resides within a hole at the bottom of a wall. The mouse leaves the hole and crawls horizontally against the wall such that its displacement from the hole is given by the function $f(t) = t^2 e^{-t}$, where time $t \geq 0$ is measured in seconds and $f(t)$ is measured in feet.

(a) Show that $f'(t) = (2t - t^2) e^{-t}$ and $f''(t) = (t^2 - 4t + 2) e^{-t}$.

(3 pts.)

- (b) Is the mouse speeding up or slowing down at $t = 1$? Justify your answer.

(2 pts.)

- (c) Determine the mouse's final displacement as $t \rightarrow \infty$.

(2 pts.)

(d) Find the maximum displacement of the mouse from the hole in the wall.

(3 pts.)

(e) A cat pursues the mouse by sneaking toward the hole in a straight line perpendicular to the mouse's path. The cat is initially 4 feet away from the hole and walks at a speed of 1 foot per second. At $t = 1$, find the rate at which the distance between the cat and mouse is changing.

(5 pts.)

25. Consider the function $g(x) = 2x^2 - 4x$.

(a) Use the limit definition of a derivative to show that $g'(x) = 4x - 4$.

(3 pts.)

(b) Using the Epsilon-Delta definition of a limit, prove the result of part (a).

(4 pts.)

(c) If $h(x) = \int_{2x}^{e^x} g(t) dt$, then calculate $h'(2)$.

(2 pts.)

(d) Find the particular solution to the initial value problem $\frac{dy}{dx} = g(x)$ with $y(0) = 2$.

(2 pts.)

(e) Use the limit of a Riemann sum to calculate $\int_0^1 g(x) \, dx$.

(4 pts.)

This marks the end of the test. The solutions and scoring rubric begin on the next page.

True/False Questions (2 points each)

1. **TRUE.** A jump discontinuity occurs when the one-sided limits both exist and equal different numbers. **
2. **TRUE.** The limit can be seen to approach 1 by L'Hôpital's Rule or by substituting $u = x^2 - 1$ to convert the limit to $\lim_{u \rightarrow 0} [(\sin u)/u] = 1$. **
3. **FALSE.** A function's values have no effect on its slope. For example, for $f(x) = 16$ and $g(x) = x^2$, we have $f(x) \geq g(x)$ on $[0, 4]$. But $f'(x) = 0$ and $g'(x) = 2x$, and $f'(x) \geq g'(x)$ does not hold on $[0, 4]$. **
4. **FALSE.** Although h is continuous on $[4, 8]$, it is not differentiable on $(4, 8)$ because a sharp turn exists at $x = 5$. **
5. **FALSE.** A function f may not have any extremum at a critical number—for example, $f(x) = x^3$ has no extremum at 0 even though $f'(0) = 0$. **
6. **TRUE.** Because f is odd, $f(-x) = -f(x)$. Thus, $\int_0^a f(-x) \, dx = -\int_0^a f(x) \, dx$. Hence,
- $$\int_{-a}^0 f(x) \, dx - \int_0^a f(-x) \, dx = \int_{-a}^0 f(x) \, dx + \int_0^a f(x) \, dx = \int_{-a}^a f(x) \, dx,$$
- which equals 0 by symmetry. **
7. **FALSE.** Part II of the Fundamental Theorem of Calculus is not applicable, as $1/x^4$ is not continuous on $[-2, 2]$ due to a vertical asymptote at $x = 0$. **

8. **TRUE.** Taking derivatives, we see

$$g'(x) = 8\cos(4x - \pi),$$

$$g''(x) = -32\sin(4x - \pi),$$

$$g'''(x) = -128\cos(4x - \pi),$$

$$\vdots$$

where each successive derivative has 4 times the amplitude of the preceding derivative. Hence, the n th derivative of g has an amplitude 4^n times greater than g . **

9. **TRUE.** Using trapezoids to approximate a positive, concave-up function captures excess area. As a numerical example, the area under the curve $y = x^2$ (which is concave up), $0 \leq x \leq 1$, is

$$\int_0^1 x^2 dx = \frac{1}{3}.$$

The Trapezoidal Rule with $n = 1$ gives the estimate

$$\frac{1}{2} (0^2 + 1^2) (1 - 0) = \frac{1}{2}. \quad \text{**}$$

10. **FALSE.** When an initial guess is selected near a horizontal tangent, the resulting tangent line has a near-0 slope. This line strikes the x -axis at a distant point, reducing the accuracy in the estimate. **

11. **FALSE.** With $u = e^x + 2$, we have $e^x = u - 2$ and $du = e^x dx = (u - 2) dx$. In addition, when $x = \ln 2$, $u = 4$; when $x = \ln 4$, $u = 6$. Hence,

$$\int_{\ln 2}^{\ln 4} \frac{e^x}{e^x + 2} dx = \int_4^6 \frac{u-2}{u} \left(\frac{1}{u-2} \right) du = \int_4^6 \frac{1}{u} du.$$

The presented statement neglected to substitute the differential dx . **

Short Questions (3 points each)

12. As $x \rightarrow \infty$ or $x \rightarrow -\infty$, the graph of $y = \frac{4x^3}{x^3 - x}$ matches the behavior of

$$y = \frac{4x^3}{x^3} = 4$$

(where solely the highest-degree terms are considered). Hence, the horizontal asymptote is

$$\boxed{y = 4}$$

Factoring, we see

$$y = \frac{4x^3}{x(x+1)(x-1)} = \frac{4x^2}{(x+1)(x-1)}, \quad x \neq 0.$$

Vertical asymptotes occur when the denominator equals 0 (and the numerator is nonzero)—that is, at

$$\boxed{x = -1}$$

$$\boxed{x = 1}$$

13. The derivative of $\sinh$ is $\cosh$, so using the Chain Rule,

$$f'(x) = 2 \sinh x \cosh x - \frac{1}{(2x)^2 + 1} (2) = 2 \sinh x \cosh x - \frac{2}{4x^2 + 1}.$$

In addition, since $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$,

$$\sinh 1 \cosh 1 = \left(\frac{e^1 - e^{-1}}{2} \right) \left(\frac{e^1 + e^{-1}}{2} \right) = \frac{1}{4} (e^1 - e^{-1}) (e^1 + e^{-1}) = \frac{1}{4} (e^2 - e^{-2}).$$

Thus,

$$f'(1) = 2 \cdot \frac{1}{4} (e^2 - e^{-2}) - \frac{2}{5} = \boxed{\frac{1}{2} (e^2 - e^{-2}) - \frac{2}{5}}$$

14. First, assuming g is continuous at 1, we have

$$-\frac{4}{(1)^2 + p} = \frac{1}{2} (2 - 1)^2 + q$$

$$-\frac{4}{1 + p} = \frac{1}{2} + q.$$

In addition,

$$g'(x) = \begin{cases} \frac{8x}{(x^2 + p)^2} & x \geq 1 \\ 2(2x - 1) & x < 1. \end{cases}$$

*

We require the one-sided derivatives to be equal at 1, so

$$\frac{8}{(1+p)^2} = 2(2-1)$$

$$\implies p = 1.$$

Then

$$-\frac{4}{1+1} = \frac{1}{2} + q \implies q = -\frac{5}{2}.$$

Therefore,

$$\boxed{p = 1} \quad \boxed{q = -\frac{5}{2}}$$

*

15. Taking derivatives, we see

$$p'(x) = 5x^{3/2} - 15x^{1/2},$$

$$p''(x) = \frac{15}{2}x^{1/2} - \frac{15}{2}x^{-1/2}.$$

*

The function p is decreasing when $p'(x) < 0$:

$$5x^{3/2} - 15x^{1/2} < 0$$

$$5x^{1/2}(x-3) < 0,$$

*

which is satisfied by $x \in (0, 3)$. In addition, p is concave up when $p''(x) > 0$:

$$\frac{15}{2}x^{1/2} - \frac{15}{2}x^{-1/2} > 0$$

$$\frac{15}{2}x^{-1/2}(x-1) > 0,$$

which is satisfied by $x \in (1, \infty)$. Hence, p is decreasing and concave up on

$$(0, 3) \cap (1, \infty) = \boxed{(1, 3)}$$

*

16. With $n = 4$, $\Delta x = \frac{9-1}{4} = 2$. By Simpson's Rule,

$$\begin{aligned}\int_1^9 \frac{2}{\sqrt{x}} dx &\approx \frac{2}{3} \left(\frac{2}{\sqrt{1}} + 4 \cdot \frac{2}{\sqrt{3}} + 2 \cdot \frac{2}{\sqrt{5}} + 4 \cdot \frac{2}{\sqrt{7}} + \frac{2}{\sqrt{9}} \right) \\ &= \boxed{\frac{16}{9} + \frac{16}{3\sqrt{3}} + \frac{8}{3\sqrt{5}} + \frac{16}{3\sqrt{7}}}\end{aligned}$$

17. The base $\frac{9t-4t^2}{1-4t^2}$ approaches 1 as $t \rightarrow \infty$. The limit is therefore in the indeterminate form 1^∞ . If

$$y = \left(\frac{9t-4t^2}{1-4t^2} \right)^t, \text{ then}$$

$$\ln y = t \ln \left(\frac{9t-4t^2}{1-4t^2} \right) = \frac{\ln \left(\frac{9t-4t^2}{1-4t^2} \right)}{1/t}.$$

Now, with L'Hôpital's Rule,

$$\begin{aligned}\lim_{t \rightarrow \infty} \ln y &= \lim_{t \rightarrow \infty} \frac{\ln \left(\frac{9t-4t^2}{1-4t^2} \right)}{1/t} \\ &= \lim_{t \rightarrow \infty} \frac{\frac{1-4t^2}{9t-4t^2} \cdot \frac{(9-8t)(1-4t^2) - (-8t)(9t-4t^2)}{(1-4t^2)^2}}{-1/t^2} \\ &= \lim_{t \rightarrow \infty} \frac{-9t^2 + 8t^3 - 36t^4}{(9t-4t^2)(1-4t^2)} \\ &= -\frac{36}{16} = -\frac{9}{4}.\end{aligned}$$

Thus,

$$\lim_{t \rightarrow \infty} \left(\frac{9t-4t^2}{1-4t^2} \right)^t = \boxed{e^{-9/4}}$$

18. Revenue is the product of quantity and price. Let s be the number of dollars by which the food stand raises the quesadilla's price. Then the price is $(4+s)$ and the quantity is $(20 - \frac{4}{2}s) = (20 - 2s)$, so the revenue is

$$R(s) = (4+s)(20-2s) = 80 + 12s - 2s^2.$$

Differentiating yields

$$R'(s) = 12 - 4s,$$

and the solution to $R'(s) = 0$ is $s = 3$. This value corresponds to the absolute maximum of R , so the food stand should raise the price by \$3. In addition, the maximum revenue is *

$$R(3) = \boxed{\$98}$$

19. Substituting $u = x + 1$ gives $x = u - 1$ and $du = dx$. In addition, when $x = 0$, $u = 1$; when $x = 7$, $u = 8$. Thus, the integral becomes

$$\begin{aligned} \int_1^8 \frac{u-1}{u^{1/3}} du &= \int_1^8 (u^{2/3} - u^{-1/3}) du \\ &= \left(\frac{3}{5} u^{5/3} - \frac{3}{2} u^{2/3} \right) \Big|_1^8 \\ &= \boxed{\frac{141}{10}} \end{aligned}$$

20. Integrating $a(t)$ gives the velocity function:

$$v(t) = -2 \cos 2t + 4 \sin 2t + C$$

for some constant C . Because $v(0) = 0$, substituting shows

$$v(0) = -2 \cos 0 + 4 \sin 0 + C = 0 \implies C = 2.$$

Accordingly, we have

$$v(t) = -2 \cos 2t + 4 \sin 2t + 2.$$

We obtain the position function by integrating $v(t)$:

$$x(t) = -\sin 2t - 2 \cos 2t + 2t + D$$

for some constant D . Substituting the initial condition $x(0) = 1$ gives

$$x(0) = 0 - 2 + 0 + D = 1 \implies D = 3.$$

Hence, the displacement as a function of time is

$$x(t) = \boxed{-\sin 2t - 2 \cos 2t + 2t + 3}$$

21. Computations are greatly simplified by using Logarithmic Differentiation. If $y = \frac{e^{-x}\sqrt{2x+7}\sec^6 4x}{x^4 \tan(2x+9)}$, then

$$\begin{aligned}\ln y &= \ln \left(\frac{e^{-x}\sqrt{2x+7}\sec^6 4x}{x^4 \tan(2x+9)} \right) \\ &= \ln(e^{-x}) + \ln(2x+7)^{1/2} + \ln(\sec^6 4x) - \ln(x^4) - \ln[\tan(2x+9)] \\ &= -x + \frac{1}{2}\ln(2x+7) + 6\ln(\sec 4x) - 4\ln x - \ln[\tan(2x+9)].\end{aligned}$$

Differentiating both sides implicitly, we obtain

$$\frac{1}{y} \frac{dy}{dx} = -1 + \frac{1}{2x+7} + 24 \tan 4x - \frac{4}{x} - \frac{2 \sec^2(2x+9)}{\tan(2x+9)}.$$

Multiplying both sides by $y = \frac{e^{-x}\sqrt{2x+7}\sec^6 4x}{x^4 \tan(2x+9)}$ produces

$$\frac{dy}{dx} = \left(\frac{e^{-x}\sqrt{2x+7}\sec^6 4x}{x^4 \tan(2x+9)} \right) \left(-1 + \frac{1}{2x+7} + 24 \tan 4x - \frac{4}{x} - \frac{2 \sec^2(2x+9)}{\tan(2x+9)} \right)$$

22. The area of the shaded region is

$$A = \pi(3r)^2 - \pi r^2 = 8\pi r^2.$$

Differentiating both sides shows

$$\frac{dA}{dr} = 16\pi r$$

$$dA = 16\pi r dr.$$

We take $dA \approx \Delta A$ and $dr \approx \Delta r$. With $r = 2$ in and $\Delta r \approx 0.1$ in, we have an estimated change in area of

$$\Delta A \approx 16\pi(2)(0.1) = \boxed{3.2\pi \text{ in}^2}$$

Long Questions (15 points each)

23. (a) Differentiating both sides of $x^2 + 4x + \frac{1}{2}y^2 = 12$ yields

$$2x + 4 + y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \boxed{\frac{-2x - 4}{y}}$$

- (b) At $x = 1$ and $y = \sqrt{14}$, the derivative is

$$\left. \frac{dy}{dx} \right|_{x=1, y=\sqrt{14}} = \frac{-6}{\sqrt{14}}.$$

Thus, an equation of the tangent line at $(1, \sqrt{14})$ is

$$\boxed{y - \sqrt{14} = \frac{-6}{\sqrt{14}}(x - 1)}$$

An equation of the normal line is

$$\boxed{y - \sqrt{14} = \frac{\sqrt{14}}{6}(x - 1)}$$

(The normal line is perpendicular to the tangent line, so their slopes are negative reciprocals of each other.)

(c) The second derivative is

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{-2x-4}{y} \right) \\ &= \frac{(-2)y - (-2x-4) \frac{dy}{dx}}{y^2} \\ &= \frac{(-2)y - (-2x-4) \left(\frac{-2x-4}{y} \right)}{y^2} \\ &= \boxed{\frac{-2y^2 - (2x+4)^2}{y^3}}\end{aligned}$$

(d) Region K is the upper half of the curve for $-6 \leq x \leq 0$. Solving for the positive solution of y in $x^2 + 4x + \frac{1}{2}y^2 = 12$ produces $y = \sqrt{24 - 2x^2 - 8x}$. Accordingly, the area of K equals

$$A = \boxed{\int_{-6}^0 \sqrt{24 - 2x^2 - 8x} dx}$$

With $n = 3$, it follows that $\Delta x = \frac{0 - (-6)}{3} = 2$. Thus, we evaluate $y = \sqrt{24 - 2x^2 - 8x}$ at the left endpoints of the subintervals $[-6, -4]$, $[-4, -2]$, and $[-2, 0]$, as follows:

$$\begin{aligned}A &\approx \left(\sqrt{24 - 2(-6)^2 - 8(-6)} + \sqrt{24 - 2(-4)^2 - 8(-4)} + \sqrt{24 - 2(-2)^2 - 8(-2)} \right) (2) \\ &= \boxed{2\sqrt{24} + 2\sqrt{32}}\end{aligned}$$

(e) Note that the curve is symmetric about $x = -2$. Accordingly, for $0 < x < 2$ if (x, y) lies on the curve, then so does the point $(-x - 4, y)$. Hence, considering positive x and y , the four vertices of an inscribed rectangle are (x, y) , $(x, -y)$, $(-x - 4, y)$, and $(-x - 4, -y)$. The rectangle therefore has a width of $[x - (-x - 4)] = 2x + 4$ and a height of $2y$, so its area is

$$A = 2y(2x + 4) = (4x + 8)\sqrt{24 - 2x^2 - 8x}.$$

Differentiating gives

$$\begin{aligned}\frac{dA}{dx} &= 4\sqrt{24-2x^2-8x} + (4x+8)\frac{-4x-8}{2\sqrt{24-2x^2-8x}} \\ &= 4\sqrt{24-2x^2-8x} + (2x+4)\frac{-4x-8}{\sqrt{24-2x^2-8x}}.\end{aligned}$$

Equating the derivative to 0 and solving, we see

$$4\sqrt{24-2x^2-8x} + (2x+4)\frac{-4x-8}{\sqrt{24-2x^2-8x}} = 0$$

$$4(24-2x^2-8x) + (2x+4)(-4x-8) = 0$$

$$x^2 + 4x - 4 = 0$$

$$\implies x = \frac{-4 \pm \sqrt{32}}{2} = -2 \pm 2\sqrt{2}.$$

Because $0 < x < 2$, we choose the positive solution, $x = -2 + 2\sqrt{2}$, which corresponds to the absolute maximum of $A(x)$ because $A(0) = A(2) = 0$. Hence, the width that maximizes the rectangle's area is

$$2(-2 + 2\sqrt{2}) + 4 = \boxed{4\sqrt{2}}$$

24. (a) By the Product Rule,

$$f'(t) = 2te^{-t} + t^2(-e^{-t}) = \boxed{(2t - t^2)e^{-t}}$$

Likewise,

$$f''(t) = (2 - 2t)e^{-t} + (2t - t^2)(-e^{-t}) = \boxed{(t^2 - 4t + 2)e^{-t}}$$

- (b) We see

$$f'(1) = e^{-1} \quad f''(1) = -e^{-1}.$$

Since $f'(1)$ and $f''(1)$ have different signs, the mouse is slowing down at $t = 1$.

- (c) The final displacement is

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} t^2 e^{-t} = \lim_{t \rightarrow \infty} \frac{t^2}{e^t},$$

which is in the indeterminate form ∞/∞ . By L'Hôpital's Rule, the limit becomes

$$\lim_{t \rightarrow \infty} \frac{2t}{e^t} = \lim_{t \rightarrow \infty} \frac{2}{e^t} = \boxed{0}$$

So the mouse approaches the hole as $t \rightarrow \infty$.

- (d) Solving $f'(t) = 0$ yields the critical numbers of f :

$$\begin{aligned} (2t - t^2)e^{-t} &= 0 \\ \implies t &= 0, 2. \end{aligned}$$

The domain of t in context is $[0, \infty)$; note that $f(0) = 0$, $f(2) = 4e^{-2}$, and $\lim_{t \rightarrow \infty} f(t) = 0$ [from part (c)]. The maximum displacement is therefore

$$\boxed{\frac{4}{e^2}}$$

- (e) Let x be the horizontal distance between the hole and the mouse, and let y be the vertical distance between the cat and the hole. If r is the distance between the cat and the mouse, then the Pythagorean Theorem gives $x^2 + y^2 = r^2$. Differentiating both sides gives

$$\begin{aligned} 2x \frac{dx}{dt} + 2y \frac{dy}{dt} &= 2r \frac{dr}{dt} \\ \implies \frac{dr}{dt} &= \frac{1}{r} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right). \end{aligned}$$

At $t = 1$ sec, $x = f(1) = e^{-1}$ and $\frac{dx}{dt} = f'(1) = e^{-1}$. In addition, at $t = 1$ sec the cat (initially 4 feet away) has traveled 1 foot, so $y = 3$ and

$$r = \sqrt{(e^{-1})^2 + 3^2} = \sqrt{9 + e^{-2}}.$$

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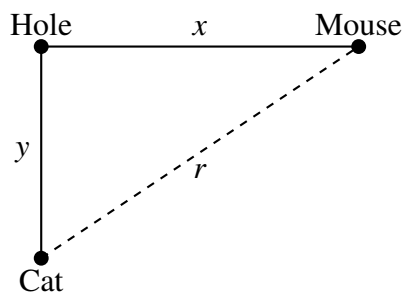
Note that the cat *approaches* the hole at a rate of 1 ft/sec, so $\frac{dy}{dt} = -1$. Hence,

$$\left. \frac{dr}{dt} \right|_{t=1} = \frac{1}{\sqrt{9 + e^{-2}}} (e^{-1} \cdot e^{-1} + 3(-1))$$

*

$$= \boxed{\frac{e^{-2} - 3}{\sqrt{9 + e^{-2}}}}$$

*



25. (a) By the definition of a derivative,

$$\begin{aligned}
 g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2(x+h)^2 - 4(x+h) - (2x^2 - 4x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{4x} - 4h - \cancel{2x^2} + \cancel{4x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4xh + 2h^2 - 4h}{h} \\
 &= \lim_{h \rightarrow 0} (4x + 2h - 4) \\
 &= \boxed{4x - 4}
 \end{aligned}$$

(b) The limit $\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = 4x - 4$ means for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$\left| \frac{g(x+h) - g(x)}{h} - (4x - 4) \right| < \varepsilon \quad \text{if} \quad 0 < |h - 0| < \delta.$$

From part (a), $\frac{g(x+h) - g(x)}{h} = 4x + 2h - 4$, so we have

$$|4x + 2h - 4 - (4x - 4)| < \varepsilon \quad \text{if} \quad 0 < |h| < \delta,$$

that is,

$$|2h| < \varepsilon \quad \text{if} \quad 0 < |h| < \delta.$$

Comparing the two inequalities, we can select

$$\delta = \frac{\varepsilon}{2}.$$

Constructing the Proof. For $0 < |h| < \delta$, we see

$$\left| \frac{g(x+h) - g(x)}{h} - (4x - 4) \right| = |2h| = 2|h| < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

Thus,

$$\left| \frac{g(x+h) - g(x)}{h} - (4x - 4) \right| < \varepsilon \quad \text{if} \quad 0 < |h| < \frac{\varepsilon}{2}.$$

(c) By Part I of the Fundamental Theorem of Calculus (with the Chain Rule),

$$h'(x) = g(e^x)(e^x) - g(2x)(2) = e^x g(e^x) - 2g(2x).$$

Hence,

$$\begin{aligned} h'(2) &= e^2 g(e^2) - 2g(4) \\ &= e^2 (2e^4 - 4e^2) - 2(16) \\ &= \boxed{2e^6 - 4e^4 - 32} \end{aligned}$$

(d) The general solution to the differential equation $\frac{dy}{dx} = 2x^2 - 4x$ is given by integration:

$$y = \int (2x^2 - 4x) dx = \frac{2}{3}x^3 - 2x^2 + C$$

for some constant C . Substituting the initial condition $y(0) = 2$, we see

$$y = 0 - 0 + C = 2 \implies C = 2.$$

Hence, the particular solution is

$$\boxed{y = \frac{2}{3}x^3 - 2x^2 + 2}$$

(e) Using a right Riemann sum with subintervals of equal width $\Delta x = \frac{1-0}{n} = \frac{1}{n}$, we have $x_i = 0 + i\Delta x =$

$\frac{i}{n}$. Thus,

$$\begin{aligned}
 \int_0^1 g(x) \, dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n g(x_i) \Delta x \\
 \int_0^1 (2x^2 - 4x) \, dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[2 \left(\frac{i}{n} \right)^2 - 4 \left(\frac{i}{n} \right) \right] \frac{i}{n} \quad * \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\frac{2i^2}{n^2} - \frac{4i}{n} \right] \frac{i}{n} \\
 &= \lim_{n \rightarrow \infty} \left(\frac{2}{n^3} \sum_{i=1}^n i^2 - \frac{4}{n^2} \sum_{i=1}^n i \right) \quad * \\
 &= \lim_{n \rightarrow \infty} \left(\frac{2}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{4}{n^2} \cdot \frac{n(n+1)}{2} \right) \quad * \\
 &= \lim_{n \rightarrow \infty} \left(\frac{(n+1)(2n+1)}{3n^2} - \frac{2(n+1)}{n} \right) \\
 &= \frac{2}{3} - 2 = \boxed{-\frac{4}{3}} \quad *
 \end{aligned}$$